Graphs, hybrids and something else

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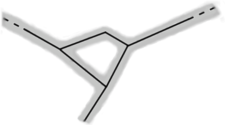
Quantum graphs



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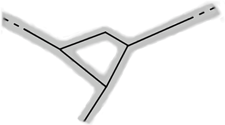

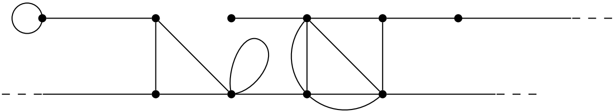

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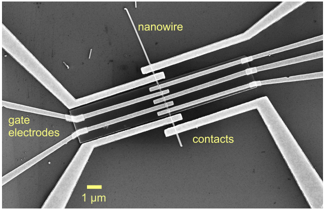
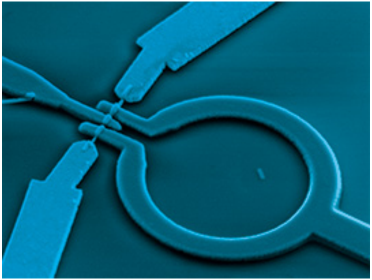
Quantum hybrids





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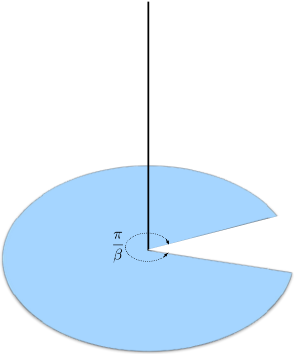

Standard NLS

The standard nonlinear Schrödinger equation (a.k.a. NLS) is

$$\imath \frac{\partial \psi}{\partial t} = -\Delta \psi + \beta |\psi|^{2\sigma} \psi,$$

with $\psi(t, \mathbf{x}) : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{C}$, $\beta \in \mathbb{R}$ and $\sigma > 0$.

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Applications:

(i) laser beams: e.g. [Rasmussen, Rypdal, Phys. Scr. '86];

(ii) Bose-Einstein condensates (a.k.a. BEC): e.g. [Dalfovo, Giorgini, Pitaevskii, Stringari, RevModPhys '99];

(iii) other applications: e.g. [Malomed, '05]:

- $\rightsquigarrow$ nonlinear optics, plasma waves, FitzHugh-Nazumo model,...

Effective equation

A system of N quantum particles with positions $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^3$ is described by a wave function $\Psi(t, \mathbf{x}_1, \ldots, \mathbf{x}_N)$ that satisfies

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$\uparrow$

Kinetic energy
of the particles

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Two-body interaction potential

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Scaling factors connected
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For a wide class of V and for suitable $a(\cdot)$ and $b(\cdot)$ and for large N,

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Hot topic in mathematical physics:

$\rightsquigarrow$ Adami, Bardos, Brennecke, Erdos, Frank Golse, Lewin, Lieb, Loss, Paul, Pickl, Rodnianski, Rougerie, Schlein, Seiringer, Solovej, Sphon, Teta, Teufel, Yau, Yngvason...

Concentrated nonlinearity

Question:

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Applications:

(i) Solid state physics: charge accumulation in semiconductors in presence of an impurity;

(ii) Nonlinear optics: propagation in presence of localized defects.

⇝ [Jona-Lasinio et al., PRB '91], [Malomed, Azbel, PRB '93], [Jona-Lasinio et al., APHY '95], [Bulashenko et al., PRB '96], [Sukhorukov et al., PRE '99] ...

Nonlinear Dirac equation on graphs with localized nonlinearities: bound states and nonrelativistic limit

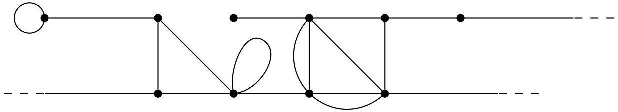
Notation

A metric graph is a graph $\mathcal{G} := (\mathrm{V}, \mathrm{E})$ s.t.:

i) $\mathcal{G}$ is a multigraph (i.e., self-loops, multiple edges, etc...);

ii) each edge $e \in \mathrm{E}$ is associated with $I_e = [0, \ell_e]$, if bounded, or with $I_e = [0, \infty)$, if unbounded (a half-line).

References: [Exner, Keating, Kuchment, Sunada, Teplyaev, '08], [Post, '12], [Berkolaiko, Kuchment, '13].



19 edges (2 self-loops, 2 multiple, 3 unbounded),
13 vertices (1 of degree two, 3 at infinity).

Note: for bounded edges the orientation of the parametrization $x_e \in I_e$ is free, while for half-lines vertices at infinity correspond to $x_e = +\infty$.

Notation

Further assumptions:

i) the cardinality of V and E is finite $\rightsquigarrow$ no periodic graphs!;

ii) $\mathcal{G}$ is connected (a path between each pair of vertices);

iii) $\mathcal{G}$ is noncompact $\rightsquigarrow$ from i) this entails at least a half-line.

As usual, a function $u : \mathcal{G} \to \mathbb{C}$ is a family of functions $u = (u_e)_{e \in \mathrm{E}}$, with $u_e := u_{|_{I_e}} : I_e \to \mathbb{C}$. As a consequence,

Lebesgue: $L^p(\mathcal{G}) := \bigoplus_{e \in \mathrm{E}} L^p(I_e) \quad \rightsquigarrow \quad \|u\|_{L^p(\mathcal{G})}^p := \|u_e\|_{L^p(I_e)}^p$

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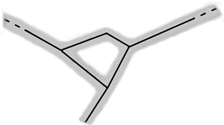
Note: usually in the definition of $H^1(\mathcal{G})$ there is also a global continuity

condition; for our purposes it is better to keep this condition separated.

Motivation

Supposed to be good approximations for constrained dynamics in which transversal dimensions are small with respect to longitudinal ones.



Topical example: NonLinear Schrödinger Equation (NLSE).

$\rightsquigarrow$ Effective model for Bose-Einstein Condensates (BEC) in ramified traps, for nonlinear optical fibers, etc...

Literature:

$\rightsquigarrow$ [Gnutzmann, Smilanski, AdvPhys '06]

$\rightsquigarrow$ [Noja, RSTA '14], [Adami, Serra, Tilli, RivMatUnivParma '17]

$\rightsquigarrow$ [Lorenzo et al., PHYSLETA '14]

NLSE with Kirchhoff conditions

The focusing (NLSE) on metric graphs with homogeneous Kirchhoff vertex conditions reads:

$$\imath \frac{\partial \psi}{\partial t} = -\Delta \psi - |\psi|^{p-2}\psi \qquad \text{on } \mathcal{G} \qquad (p \geq 2) \qquad \text{(NLSE)}$$

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where "$e \succ \mathrm{v}$" means that e is incident at v and $\frac{dv_e}{dx_e}(\mathrm{v})$ stands for $v_e'(0)$ or $-v_e'(\ell_e)$ depending on the orientation of I_e.

Bound states of the NLSE

Problem: existence of bound states (B.S.), i.e. L^2-solutions of the form

$$\psi(t,x) := e^{-\imath\lambda t}u(x), \qquad \lambda \in \mathbb{R}.$$

Definition – B.S. of the NLSE

A bound state of the (NLSE) is a function $u \not\equiv 0$ s.t. $u \in \mathrm{dom}(-\Delta)$ and there exists $\lambda \in \mathbb{R}$ s.t.

$$-u_e'' - |u_e|^{p-2}\,u_e = \lambda u_e, \qquad \forall e \in \mathrm{E}.$$

Literature:

i) real line: e.g. [Zakharov, Shabat, JETP '72], [Cazenave, Lions, CMP '82];

ii) infinite N-star: e.g. [Adami, Cacciapuoti, Finco, Noja, JPA '12 - JDE '14 - ANIHPC '14], [Kairzhan, Pelinovsky, JPA '18 - JDE '18];

ii) tadpole: e.g. [Cacciapuoti, Finco, Noja, PhysRevE '15], [Noja, Pelinovsky, Shaikhova, Nonlin '15];

iv) general: e.g. [Adami, Serra, Tilli, CVPDE '15 - JFA '16 - CMP '17 - arXiv '17].

Localized nonlinearity

Definition – Compact core

The compact core of $\mathcal{G}$, denoted by $\mathcal{K}$, is the metric subgraph of $\mathcal{G}$ consisting of all its bounded edges.

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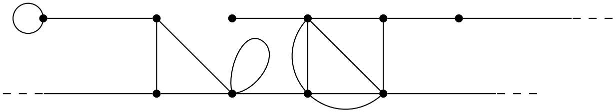

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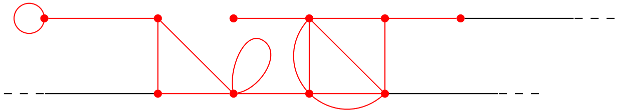

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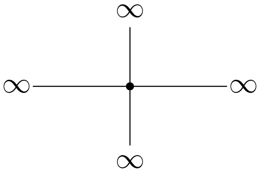

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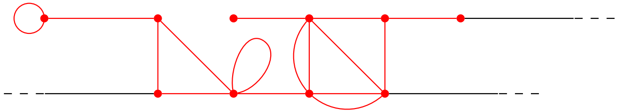

Problem: what happens when the nonlinearity affects only $\mathcal{K}$?

([Gnutzmann, Smilanski, Derevyanko, PhysRevA '11], [Noja, RSTA '14])

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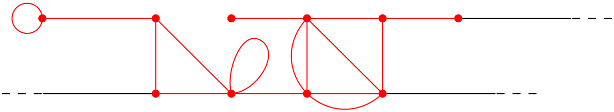

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Definition – B.S. of the NLSE with Localized Nonlinearity

A bound state of the (NLSE) with Localized Nonlinearity (L.N.) is a function $u \not\equiv 0$ s.t. $u \in \mathrm{dom}(-\Delta)$ and there exists $\lambda \in \mathbb{R}$ s.t.

$$-u_e'' - \chi_{\mathcal{K}} |u_e|^{p-2} u_e = \lambda u_e, \qquad \forall e \in \mathrm{E}.$$

B.S. with Localized Nonlinearity

1. [Tentarelli, JMAA '16]:
   - ⇝ Existence/nonexistence of constrained minimizers of

$$\mathcal{E}_{\mathcal{K}}(v) := \frac{1}{2}\int_{\mathcal{G}} |v'|^2\,dx - \frac{1}{p}\int_{\mathcal{K}} |v|^p\,dx,$$

   on $\left\{\|v\|^2_L(\mathcal{G}) = \mu > 0\right\}$, in the L^2-subcritical case $p \in (2,6)$.

2. [Serra,Tentarelli, JDE '16], [Serra,Tentarelli, NA '16]:
   - ⇝ Existence/nonexistence (respectively) of constrained critical points of the functional $\mathcal{E}_{\mathcal{K}}(\,\cdot\,)$ (in the L^2-subcritical case).

3. [Dovetta,Tentarelli, arXiv '18]:
   - ⇝ Existence/nonexistence of constrained minimizers of $\mathcal{E}_{\mathcal{K}}(\,\cdot\,)$ in the L^2-critical case $p = 6$ (for a tadpole graph);
   - ⇝ Ongoing project.

B.S. with Localized Nonlinearity

	Exponents	Ground	Bound
NLSE	$p \in (2,4)$	- yes, $\forall \mu > 0$	(see box below)
	$p \in [4,6)$	- yes if $\mu > \mu_1$ - no if $\mu < \mu_2$ - unknown if $\mu \in [\mu_2, \mu_1]$	- yes (and multiple) if μ is large enough - yes if $\mathcal{G}$ has a loop or two terminal edges - no (with $\lambda \leq 0$) if $\mu < (p/2)^{2/(2-p)}\mu_2$ - no (with $\lambda \geq 0$) if $\mathcal{G}$ has (at most) one terminal edge and no loops - unknown otherwise
	$p = 6$	- yes if $\mu \in [\mu_{\mathcal{K}}, \mu_{\mathbb{R}}]$ and if no terminal edges and no cycle coverings* - no otherwise	- yes if $\mathcal{G}$ has a loop or two terminal edges - no (with $\lambda \geq 0$) if $\mathcal{G}$ has (at most) one terminal edge and no loops - unknown otherwise
	$p > 6$	- unknown	(see box above)

From Schrödinger to Dirac

Recenly, [Sabirov, Babajanov, Matrasulov, Kevrekidis, arXiv '17] proposed the study of the NonLinear Dirac Equation (NLDE)

$$\imath \frac{\partial \Psi}{\partial t} = \mathcal{D}\Psi - |\Psi|^{p-2}\Psi \qquad \text{on } \mathcal{G} \qquad (p \geq 2), \qquad \text{(NLDE)}$$

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Applications: take into account relativistic effects.

$\rightsquigarrow$ [Sabirov, Babajanov, Matrasulov, Kevrekidis, arXiv '17], [Haddad, Carr, PHYSD '09 - NJP '15].

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Spinors on metric graphs

Since $\mathcal{D}$ is a matricial operator, the first difference with (NLSE) is that (NLDE) has a spinorial nature; namely the unknown is a 2-spinor:

$$\psi = (\psi_e) = \begin{pmatrix} \varphi \\ \eta \end{pmatrix} : \mathcal{G} \longrightarrow \mathbb{C}^2$$

where $\varphi = (\varphi_e)$, $\eta = (\eta_e)$ are functions on graphs.

Lebesgue: $L^p(\mathcal{G}, \mathbb{C}^2) := \{\varphi, \eta \in L^p(\mathcal{G})\}$

$$\rightsquigarrow \quad \|\psi\|^p_{L^p(\mathcal{G},\mathbb{C}^2)} := \|\varphi\|^p_{L^p(\mathcal{G})} + \|\eta\|^p_{L^p(\mathcal{G})},$$

Sobolev: $H^1(\mathcal{G}, \mathbb{C}^2) := \{\varphi, \eta \in H^1(\mathcal{G})\}$

$$\rightsquigarrow \quad \|\psi\|^2_{H^1(\mathcal{G},\mathbb{C}^2)} := \|\varphi\|^2_{H^1(\mathcal{G})} + \|\eta\|^2_{H^1(\mathcal{G})}.$$

However, $\mathcal{D}$ is just formal since it is not defined at the vertices! ...

Kirchhoff-type vertex conditions

To find a suitable s.a. realization of $\mathcal{D}$

We choose the following:

$$\mathcal{D}\psi_{|_{I_e}} = \mathcal{D}_e\psi_e := -\imath c\sigma_1\psi_e' + mc^2\sigma_3\psi_e, \quad \forall e \in \mathrm{E}, \quad \forall \psi \in \mathrm{dom}(\mathcal{D}),$$

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$$\mathrm{dom}(\mathcal{D}) := \left\{\psi \in H^1(\mathcal{G}, \mathbb{C}^2) : \ \psi \text{ satisties (KT1)-(KT2)}\right\},$$

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with

$$\varphi_{e_1}(\mathrm{v}) = \varphi_{e_2}(\mathrm{v}), \quad \forall e_1, e_2 \succ \mathrm{v}, \quad \forall \mathrm{v} \in \mathrm{V} \backslash \mathrm{V}_\infty \tag{KT1}$$

$$\sum_{e \succ \mathrm{v}} \eta_e(\mathrm{v})_\pm = 0, \quad \forall \mathrm{v} \in \mathrm{V} \backslash \mathrm{V}_\infty \tag{KT2}$$

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$\eta_e(\mathrm{v})_\pm$ stands for $\eta_e(0)$ or $-\eta_e(\ell_e)$ depending on the orientation of I_e.

Kirchhoff-type vertex conditions

One can check (using [Bulla, Trenkler, JMP '90], [Bolte, Harrison, JPA '03],[Post, '08], [C., Malamud, Posilicano, JDE '13]) that:

i) $\mathcal{D}$ is self-adjoint;

ii) the spectrum is absolutely continuous and presents a gap, i.e.

$$\sigma(\mathcal{D}) = (-\infty, -mc^2] \cup [mc^2, +\infty).$$

We call (KT1)-(KT2) Kirchhoff-type vertex conditions.

Why?

1. They identify (as Kirchhoff for $-\Delta$) the free case: no effect at the vertices (which are then mere junctions between the edges).

   $\rightsquigarrow$ Introduced by [Sabirov, Babajanov, Matrasulov, Kevrekidis, arXiv '17]
   (where are derived by some conservation laws).

2. They converge to the Kirchhoff ones in the nonrelativistic limit.

Quadratic form

Formally, the quadratic form associated with $\mathcal{D}$ should read

$$\mathcal{Q}(\psi) := (\psi, \mathcal{D}\psi)_{L^2(\mathcal{G},\mathbb{C}^2)} = \frac{1}{2}\int_{\mathcal{G}} \psi \cdot \mathcal{D}\psi \, dx,$$

with domain $H^{1/2}(\mathcal{G},\mathbb{C}^2) := \{\varphi,\, \eta \in H^{1/2}(\mathcal{G})\}$.

i) the derivative of a function in $H^{1/2}(I_e)$ belongs to $H^{-1/2}(I_e)$, which is not the dual of $H^{1/2}(I_e)$ if I_e is bounded;

ii) $H^{1/2}(I_e) \not\hookrightarrow C^0(I_e)$ $\Rightarrow$ cannot "just add" a boundary condition

$\rightsquigarrow$ as for $-\Delta$, where the definitions of the quadratic form and its domain are immediate, i.e.

$$\frac{1}{2}\|v'\|^2_{L^2(\mathcal{G})}, \qquad \text{with domain} \quad H^1(\mathcal{G})\text{“}+\text{”(K1)}.$$

Problem: how to define $\mathcal{Q}$ and $\mathrm{dom}(\mathcal{Q})$?

Quadratic form

- ⇝ There exists U unitary that transforms $\mathcal{D}$ into a multiplication operator f on $L^2(M,\, d\mu)$.
- ⇝ Hence, $\mathcal{Q}(v) := \frac{1}{2}(Uv, f\,Uv)_{L^2(M,\,d\mu)}$ and

$$\mathrm{dom}(\mathcal{Q}) := \{\|\sqrt{|f|}Uv\|_{L^2(M,\,d\mu)} < \infty\}.$$

This could seem quite abstract, but, by standard Interpolation Theory,

$$\mathrm{dom}(\mathcal{Q}) = \left[L^2(\mathcal{G}), \mathrm{dom}(\mathcal{D})\right]_{\frac{1}{2}}$$

and

$$H^{1/2}(\mathcal{G}, \mathbb{C}^2) = \left[L^2(\mathcal{G}), H^1(\mathcal{G}, \mathbb{C}^2)\right]_{\frac{1}{2}},$$

$$\mathrm{dom}(\mathcal{Q}) \hookrightarrow H^{1/2}(\mathcal{G}, \mathbb{C}^2) \hookrightarrow L^p(\mathcal{G}, \mathbb{C}^2) \qquad (2 \le p < \infty).$$

Back to nonlinear

Now, we have a precise meaning for (NLDE)

$$\imath \frac{\partial \Psi}{\partial t} = \mathcal{D}\Psi - |\Psi|^{p-2}\Psi$$

where $\mathcal{D}$ is the Dirac operator with Kirchhoff-type vertex conditions.

Goal: bound states, that is solutions of the form

$$\Psi(t,x) := e^{-\imath\omega t}\psi(x), \qquad \omega \in \mathbb{R}.$$

Problem: we cannot search for costrained minimizers since, the associated

$$\mathcal{E}(\psi) := \mathcal{Q}(\psi) - \frac{1}{p}\int_{\mathcal{G}} |\psi|^p \, dx$$

is unbounded below, even if one fixes the L^2-norm

$\rightsquigarrow$ due to the spectral properties of $\mathcal{D}$, precisely to the presence of an infinite negative portion of the spectrum.

Main results: B.S. with localized nonlinearities

In addition, we decided to first study the case of the localized nonlinearity; that is, to search for

Definition – B.S. of the NLDE with L.N.

A B.S. of the (NLDE) with L.N. is a spinor $\psi \not\equiv 0$ s.t.:

i) $\psi \in \mathrm{dom}(\mathcal{D})$;

ii) there exists $\omega \in \mathbb{R}$ s.t.

$$\mathcal{D}_e \psi_e - \chi_{\mathcal{K}} |\psi_e|^{p-2} \psi_e = \omega \psi_e, \qquad \forall e \in \mathrm{E}.$$

Then, we proved:

Theorem 1 [T.B.C., SIMA '19]

Let $\mathcal{K} \neq \emptyset$ and let $p > 2$. Then, for every $\omega \in (-mc^2, mc^2) = \mathbb{R} \backslash \sigma(\mathcal{D})$, there exists infinitely many (distinct pairs of) B.S. of frequency ω of the (NLDE) with L.N..

Main results: nonrelativistic limit

By the definition of $\mathcal{D}$, the B.S. obtained via Theorem 1 depend on the relativistic parameter c (the mass m) and the frequency ω:

- $\rightsquigarrow$ meant as B.S. at (a fixed value) speed of light c and frequency ω.

Theorem 2 [T.B.C., SIMA'19]

Let $\mathcal{K} \neq \emptyset$, $p \in (2,6)$ and $\lambda < 0$. Let also (c_n) and (ω_n) be two real sequences such that

$$0 < c_n, \omega_n \to \infty, \qquad \omega_n < mc_n^2, \qquad \omega_n - mc_n^2 \to \frac{\lambda}{m}.$$

If $\left\{\psi_n = (\varphi_n, \eta_n)^T\right\}$ is a sequence of B.S. of frequency ω_n of the (NLDE)
with L.N. at speed of light c_n,

$$\varphi_n \to u \qquad \text{and} \qquad \eta_n \to 0 \qquad \text{in} \quad H^1(\mathcal{G}),$$

where u is a B.S. of frequency λ of the (NLSE) with L.N.

Remarks

B.S. EXISTENCE

1. First variational result on the B.S. of the (NLDE) on metric graphs.

2. Differences with respect to the (NLSE) on graphs:
   i) one cannot search for constrained minimizers of a proper energy functional, since the kinetic part $\mathcal{Q}$ is unbounded from below;
   ii) one cannot use the adaptations of direct methods of calculus of variations introduced for the (NLSE);
   iii) $\mathcal{Q}$ is strongly indefinite in sign:
      - $\rightsquigarrow$ more refined tools of Critical Point Theory;
      - $\rightsquigarrow$ more complex geometry of the functional (linking);
   iv) the spinorial nature and the implicit definition of $\mathcal{Q}$:
      - $\rightsquigarrow$ one cannot use of the techniques of rearrangements and "graph surgery" developed for (NLSE).

Remarks

3. It is necessary to adapt classical techniques for (NLDE) on standard domains (e.g. [Rabinowitz, '80], [Esteban, Séré, CMP '95], [Struwe, '08]).

NONRELATIVISTIC LIMIT

1. Actually, the B.S. converge to the (NLSE) with a pre-factor $2m$ in front of the nonlinearity.

2. Theorem 2 holds just in the L^2-subcritical case $p \in (2, 6)$.

3. As a byproduct, Theorem 2 is an existence result for the (NLSE) parametrized by λ and not by L^2-norm.

4. The meaning of the nonrelativistic limit is to investigate what occurs when the relativistic effects become negligible (i.e. $c_n \to \infty$):
   - $\rightsquigarrow$ the convergence to (NLSE) is a rigorous justification of the physical intuition.
   - $\rightsquigarrow$ also justifies Kirchhoff-type vertex conditions.